***A COMPARATIVE STUDY ON NUMERICAL SOLUTION OF INITIAL VALUE PROBLEM BY USING EULER’S METHOD, MODIFIED EULER’S METHOD AND RUNGE-KUTTA METHOD***

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***ABSTRACT****: In this article, we discussed the numerical solutions of ordinary differential equations with initial value problems by Euler’s Method, Modified Euler’s Method and Runge- Kutta Methods. Here the solutions of some numerical examples have been obtained with the help of MATLAB program as well as we determined the exact analytic solutions. In order to validate the accuracy, we compared the exact solutions with the numerical approximate solutions. We observed that when the step size is very small the accuracy of the solution become more correct. And the relative error is the difference of the approximate solution and analytic solution. Among these three proposed methods we observed that only the relative error is nominal for Runge-Kutta fourth order method.*

***KEYWORDS:*** *Initial value problem, Euler method, Modified Euler’s method, Runge-Kutta method, Error analysis.*

 ***INTRODUCTION***

**I**n mathematical modeling differential equations are generally used in the field of science and engineering. In the field of mathematical physics different problems are arise as the form of differential equations. These types of differential equations may the formation either ordinary differential equations or partial differential equations. Practically, most of the models of the problem which are formulated by means of these equations are so complicated to determine the exact solution and one of two approaches is taken to approximate the solution. The first technique is to reduce the differential equation in to one that can be solved exactly and then use the result of the reduced equation to approximate the solution to the original problem. Another technique, which we will verify in this article, uses methods of approximations the solution of original problem. This is the technique that is generally taken as the approximation methods give more perfect results and relative error information. Numerical methods are usually used for solving mathematical problems that are articulated in the field of science and engineering in which case the determination of the exact solution is so hard or impossible. Only a few numbers of differential equations can be solved analytically. Consequently, to obtain the analytical solution for differential equation there exist different methods. A huge number of differential equations are unable to determine the solution in closed form using familiar analytical methods, in which case we apply numerical technique for solving a differential equation under certain initial restriction or restrictions. There exist different kinds of practical numerical methods for finding the solution of initial value problem of ordinary differential equations. In this research article we present three outstanding numerical methods such as Euler’s method, Modified Euler’s method and Runge-Kutta method for finding the solution of initial value problems for ordinary differential equations.

 From the literature analysis, we may understand that numerous works in numerical solutions of initial value problems applying Euler’s method, Modified Euler’s method and Runge-Kutta method have been performed. Various authors have endeavored to solve initial value problems to determine high correctness promptly by using several methods, such as Euler’s method, Modified Euler’s method and Runge-Kutta method and so on. In [1] the author deliberated the accurate solution of initial value problems for ordinary differential equation by using Runge-Kutta fourth order method. In [2-3] discussed on accurate solution of initial value problems for ordinary differential equations. [3] considered on some numerical methods for solving initial value problems in ordinary differential equations. Also [4]-[16] studied numerical solutions of initial value problems for ordinary differential equations using a variety of numerical methods. In this work Euler method, Euler modified method and Runge Kutta method are performed without any discretization, alteration or limiting assumption for solving initial value problems of ordinary differential equations. The Euler method is conventionally the first numerical method. It is extraordinarily easy to realize and geometrically simple to expressive however not very handy; the technique has restricted accuracy for further intricate functions. A more vigorous and complicated numerical method is the Runge Kutta method. This method is the most commonly used one since it provides consistent initial values and is especially suitable when the calculation of higher derivatives are intricate. The numerical outcomes are very hopeful. As a final point, a pair of example including different kinds of ordinary differential equations is given to validate the anticipated formula. The solutions of each numerical illustration specify that the convergence and error analysis which are discussed demonstrate the effectiveness of the methods. In case of Euler method, Euler modified method to determine the solution of the differential equation numerically is less resourceful since it requires h to be tiny for finding logical accuracy. However in Runge-Kutta method, the derivatives of superior order are not prerequisite and they are intended to provide better accuracy with the benefit of requiring only the functional values at some preferred points on the sub-interval. Runge- Kutta method is a more common and spontaneous method as compared to that of the Euler method, Euler modified method. We detect that in the Euler method, Euler modified method extremely small step size converges to analytical solution. So, more number of approximation is needed. On the contrary, Runge Kutta method gives improved results and it converges closer to analytical solution and has less iteration to get exactitude solution. This article is rearranged as follows: segment 2: problem formulations; segment 3: error analysis; segment 4: numerical examples; segment 5:

discussion of results; and the final segment: the conclusion of the article.

***SEGMENT:2 Problem Formulation***

In this segment we consider three numerical methods for obtaining the numerical solutions of the initial value problem (IVP) of the first-order ordinary differential equation is of the form

 (1)

where and is the given function and is the solution of the equation (1). In this article, we find the solution of this equation on a finite interval , starting with the initial point . A continuous approximation to the solution will not be obtained; instead, approximations to *y* will be generated at various values, called mesh points, in the interval . Numerical methods employ the Equation (1) to obtain approximations to the values of the solution corresponding to various selected values of the solution corresponding to different selected values of . The parameter h is called the step size. The numerical solution of (1) is given by a set of points and each point is an approximation to the corresponding point {} on the solution curve.

***2.1 Euler Method***

Euler’s method is the simplest one-step method. It is basic explicit method for numerical integration of ordinary differential equations. Euler proposed his method for initial value problems (IVP) in 1768. It is first numerical method for solving IVP and serves to illustrate the concepts involved in the advanced methods. It is important to study because the error analysis is easier to understand. The general formula for Euler approximation is

;

***2.2 Modified Euler Method***

In this method the curve in the interval where is approximated by the line through with the slope, which is the slope as the middle point whose abscissa is average of and . A generalized form of Euler’s Modified formula is

***2.3 Runge-Kutta Method***

This method was devised by two German mathematicians, Runge about 1894 and extended by Kutta a few years later. The Runge-Kutta method is most familiar because it is pretty accurate, steady and simple to program. This method is notable by their order in the logic that they concur with Taylor’s series solution up to terms of where *r* is the order of the method. It do not require previous computational of higher derivatives of as in Taylor’s series method. The fourth order Runge-Kutta method (RK4) is broadly used for solving initial value problems (IVP) for ordinary differential equation (ODE). The general formula for Runge-Kutta method is

***SEGMENT:3 Error Analysis***

There exist two kinds of errors in numerical solution of ordinary differential equations. Round-off errors and Truncation errors happen when ordinary differential equations are solved numerically. Rounding errors initiate from the verity that computers can only characterize numbers using a fixed and restricted number of important figures. Thus, such numbers cannot be represented accurately in computer memory. The inconsistency introduced by this restriction is call Round-off error. Truncation errors in numerical study occur when approximations are used to determine a number of quantities. The exactness of the solution will rely on how miniature we make the step size, *h*. A numerical method is said to be convergent if

. Where y(xn) denotes the approximate solution and yn denotes the exact solution. In this work we consider two initial value problems to examine accuracy of the proposed methods. The Approximated solution is determined by using MATLAB software for three proposed numerical methods at different step size. The maximum error is defined by

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Euler’s Method |  | Euler’s Modified Method |  | Runge-Kutta Method |  | Exact Solution |
| *xn* |  | ER |  | ER |  | ER |  |
| 0.1 | 1.0000000000000000  | 5.34652E−03 | 1.005250000000000 | 9.65218E-05 | 1.0053464802083334 | 4.16045E−08 | 1.0053465218128410 |
| 0.2 | 1.0110000000000000 | 1.18895E−02 | 1.022661643750000 | 0.000227819 | 1.0228893798037348 | 8.26716E−08 | 1.0228894624752929 |
| 0.3 | 1.0352199999999998 | 1.99720E−02 | 1.054783850254688 | 0.000408114 | 1.0551918407370900 | 1.23029E−07 | 1.0551919637660336 |
| 0.4 | 1.0752765999999998 | 3.00424E−02 | 1.104662546534985 | 0.000656406 | 1.1053187896458685 | 1.63325E−07 | 1.1053189529706604 |
| 0.5 | 1.1342876640000000 | 4.26873E−02 | 1.175976557420941 | 0.000998415 | 1.1769747667144460 | 2.05805E−07 | 1.1769749725189769 |
| 0.6 | 1.2160020472000000 | 5.86769E−02 | 1.273209735845547 | 0.001469256 | 1.2746787363539485 | 2.55624E−07 | 1.2746789919776722 |
| 0.7 | 1.3249621700320000 | 7.90261E−02 | 1.401871127660406 | 0.002117191 | 1.4039879953710888 | 3.23030E−07 | 1.4039883184007750 |
| 0.8 | 1.4667095219342400 | 1.05078E−01 | 1.568778873945045 | 0.003008896 | 1.5717873427344033 | 4.26941E−07 | 1.5717877696756601 |
| 0.9 | 1.6480462836889793 | 1.38620E−01 | 1.782428926401787 | 0.004236927 | 1.7866652528501639 | 6.00769E−07 | 1.7866658536190383 |
| 1.0 | 1.8773704492209877 | 1.82037E−01 | 2.053477058070325 | 0.005930347 | 2.0594065035273252 | 9.01815E−07 | 2.059407405342576 |

***1(a)***

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Euler’s Method |  | Euler’s Modified Method |  | Runge-Kutta Method |  | Exact Solution |
| *xn* |  | ER |  | ER |  | ER |  |
| 0.1 | 1.002625000000000 | 2.72152E−03 | 1.0052500 00000000 | 9.65218E-05 | 1.00534648 0208333 | 4.1605E-08 | 1.00534652 18128410 |
| 0.2 | 1.0168241609375002 | 6.06530E−03 | 1.022661643750000 | 0.000227819 | 1.0228893798037348 | 8.2672E-08 | 1.0228894624752929 |
| 0.3 | 1.0449798075787111 | 1.02122E−02 | 1.054783850254688 | 0.000408114 | 1.0551918407370900 | 1.2303E-07 | 1.0551919637660336 |
| 0.4 | 1.0899197085245087 | 1.53992E−02 | 1.104662546534985 | 0.000656406 | 1.1053187896458685 | 1.6332E-07 | 1.1053189529706604 |
| 0.5 | 1.1550367600056364 | 2.19382E−02 | 1.175976557420941 | 0.000998415 | 1.1769747667144460 | 2.058E-07 | 1.1769749725189769 |
| 0.6 | 1.244439027678436 | 3.02400E−02 | 1.273209735845547 | 0.001469256 | 1.2746787363539485 | 2.5562E-07 | 1.2746789919776722 |
| 0.7 | 1.3631397949603248 | 4.08485E−02 | 1.401871127660406 | 0.002117191 | 1.4039879953710888 | 3.2303E-07 | 1.4039883184007750 |
| 0.8 | 1.517300301075834 | 5.44875E−02 | 1.568778873945045 | 0.003008896 | 1.5717873427344033 | 4.2694E-07 | 1.5717877696756601 |
| 0.9 | 1.7145419864264193 | 7.21239E−02 | 1.782428926401787 | 0.004236927 | 1.7866652528501639 | 6.0077E-07 | 1.7866658536190383 |
| 1.0 | 1.9643507036668488 | 9.50567E−02 | 2.053477058070325 | 0.005930347 | 2.0594065035273252 | 9.0182E-07 | 2.059407405342576 |

***1(b)***

 ***SEGMENT: 4 Numerical Examples***

In this section we consider two numerical examples to verify which numerical methods converge faster to analytical solution. Numerical solutions and errors are computed.

***Example 1:*** We consider the initial value problem on the interval . Then the exact solution of the given problem is as . The approximate solutions and maximum errors are obtained and shown in tables 1(a)-(d).

**Table 1** (a)Numerical approximations and maximum errors for step; size h=0.1; (b) Numerical approximations and maximum errors for step size h=0.05; (c) Numerical approximations and maximum errors for step size h=0.025; (d) Numerical approximations and maximum errors for step size h=0.0125.

***1(c)***

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Euler’s Method |  | Euler’s Modified Method |  | Runge-Kutta Method |  | Exact Solution |
| *xn* |  | ER |  | ER |  | ER |  |
| 0.1 | 1.0039732143920899 | 1.37331E-03 | 1.0053404 3378088 | 6.08803E-06 | 1.0053465216505684 | 1.6228E-10 | 1.005346521812841 |
| 0.2 | 1.0198254164252103 | 3.06405E−03 | 1.022875020804634 | 1.44417E-05 | 1.0228894621535374 | 3.2176E-10 | 1.0228894624752929 |
| 0.3 | 1.050026859341876 | 5.16510E−03 | 1.055165966239071 | 2.59975E-05 | 1.0551919632892464 | 4.7679E-10 | 1.0551919637660336 |
| 0.4 | 1.097520387412045 | 7.79857E−03 | 1.105276938021313 | 4.20149E-05 | 1.105318952342068 | 6.286E-10 | 1.1053189529706604 |
| 0.5 | 1.1658497569818174 | 1.11252E−02 | 1.176910764603475 | 6.42079E-05 | 1.1769749717346203 | 7.8435E-10 | 1.1769749725189769 |
| 0.6 | 1.259321437932899 | 1.53576E−02 | 1.274584063574121 | 9.49284E-05 | 1.2746789910151588 | 9.6252E-10 | 1.2746789919776722 |
| 0.7 | 1.3832106899613061 | 2.07776E−02 | 1.403850894637291 | 0.000137424 | 1.4039883171991199 | 1.2017E-09 | 1.403988318400775 |
| 0.8 | 1.5440262079869167 | 2.77616E−02 | 1.571591569133329 | 0.000196201 | 1.5717877681003285 | 1.5753E-09 | 1.5717877696756601 |
| 0.9 | 1.7498524246222582 | 3.68134E−02 | 1.786388311742869 | 0.000277542 | 1.786665851402671 | 2.2164E-09 | 1.7866658536190383 |
| 1.0 | 2.0107951384702343 | 4.86123E−02 | 2.059017158076531 | 0.000390247 | 2.0594074019860655 | 3.3565E-09 | 2.059407405342576 |

 ***1(d)***

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Euler’s Method |  | Euler’s Modified Method |  | Runge-Kutta Method |  | Exact Solution |
| *xn* |  | ER |  | ER |  | ER |  |
| 0.1 | 1.0046566697375803 | 6.89852E−04 | 1.0053404 3378088 | 6.08803E-06 | 1.0053465218027011 | 1.01399E−11 | 1.005346521812841 |
| 0.2 | 1.0213494287582197 | 1.54003E−03 | 1.022875020804634 | 1.44417E-05 | 1.0228894624551952 | 2.00999E−11 | 1.0228894624752929 |
| 0.3 | 1.0525943319732851 | 2.59763E−03 | 1.055165966239071 | 2.59975E-05 | 1.0551919637362754 | 2.97600E−11 | 1.0551919637660336 |
| 0.4 | 1.1013943547977403 | 3.92460E−03 | 1.105276938021313 | 4.20149E-05 | 1.1053189529314784 | 3.91900E−11 | 1.1053189529706604 |
| 0.5 | 1.1713723532944522 | 5.60262E−03 | 1.176910764603475 | 6.42079E-05 | 1.1769749724701777 | 4.88001E−11 | 1.1769749725189769 |
| 0.6 | 1.2669392048911032 | 7.73979E−03 | 1.274584063574121 | 9.49284E-05 | 1.274678991917932 | 5.97400E−11 | 1.2746789919776722 |
| 0.7 | 1.3935085610750229 | 1.04798E−02 | 1.403850894637291 | 0.000137424 | 1.403988318326378 | 7.44000E−11 | 1.403988318400775 |
| 0.8 | 1.5577734062756774 | 1.40144E−02 | 1.571591569133329 | 0.000196201 | 1.571787769578305 | 9.73599E−11 | 1.5717877696756601 |
| 0.9 | 1.7680647857193632 | 1.86011E−02 | 1.786388311742869 | 0.000277542 | 1.786665853482107 | 1.36930E−10 | 1.7866658536190383 |
| 1.0 | 2.034820184163635 | 2.45872E−02 | 2.059017158076531 | 0.000390247 | 2.0594074051349014 | 2.07670E−10 | 2.059407405342576 |

***Example 2:***  We consider the initial value problem on the interval . The exact solution of the given problem is The approximate solutions and maximum errors are obtained and shown in tables 2(a)-(d).

 ***2(a)***

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Euler’s Method |  | Euler’s Modified Method |  | Runge-Kutta Method |  | Exact Solution |
| *xn* |  | ER |  | ER |  | ER |  |
| 0.1 | 0.9000000000000000 | 1.35091E−02 | 0.9145000 0000000 | 0.000990872 | 0.9135089320204528 | 1.95878E−07 | 0.913509127898782 |
| 0.2 | 0.8280000000000001 | 2.12185E−02 | 0.850700872600742 | 0.001482354 | 0.8492181710604544 | 3.47642E−07 | 0.849218518702443 |
| 0.3 | 0.7760016000000001 | 2.58218E−02 | 0.803539722296333 | 0.001716324 | 0.8018229448618213 | 4.53096E−07 | 0.8018233979576023 |
| 0.4 | 0.7390637996797441 | 2.87198E−02 | 0.769598236661347 | 0.001814651 | 0.7677830621260258 | 5.24033E−07 | 0.7677835861595071 |
| 0.5 | 0.7140048216672278 | 3.06849E−02 | 0.746530852829335 | 0.001841152 | 0.7446891282464022 | 5.72232E−07 | 0.7446897004786337 |
| 0.6 | 0.6987247742141842 | 3.21636E−02 | 0.732718569364669 | 0.001830167 | 0.730887796150559 | 6.06628E−07 | 0.7308884027785085 |
| 0.7 | 0.691826629656969 | 3.34247E−02 | 0.727051749410847 | 0.00180045 | 0.725250665872579 | 6.33400E−07 | 0.7252512992720983 |
| 0.8 | 0.6923920851827047 | 3.46350E−02 | 0.728789350891342 | 0.001762265 | 0.7270264295821635 | 6.56635E−07 | 0.7270270862176577 |
| 0.9 | 0.6998427720349557 | 3.59008E−02 | 0.737464721311611 | 0.001721133 | 0.7357429095800243 | 6.78965E−07 | 0.7357435885449581 |
| 1.0 | 0.7138506309611446 | 3.72897E−02 | 0.752820255668837 | 0.001679904 | 0.751139649932897 | 7.02025E−07 | 0.7511403519579868 |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Euler’s Method |  | Euler’s Modified Method |  | Runge-Kutta Method |  | Exact Solution |
| *xn* |  | ER |  | ER |  | ER |  |
| 0.1 | 0.9072500000000000 | 6.25913E−03 | 0.9137386 3169801 | 0.000229504 | 0.9135091213176563 | 6.58113E−09 | 0.913509127898782 |
| 0.2 | 0.8392609277701965 | 9.95759E−03 | 0.849562976531398 | 0.000344458 | 0.8492185048488676 | 1.38536E−08 | 0.849218518702443 |
| 0.3 | 0.7895884571598809 | 1.22349E−02 | 0.802223388565394 | 0.000399991 | 0.8018233782217195 | 1.97359E−08 | 0.8018233979576023 |
| 0.4 | 0.7540743267065178 | 1.37093E−02 | 0.768207590571715 | 0.000424004 | 0.7677835620326319 | 2.41269E−08 | 0.767783586159507 |
| 0.5 | 0.7299570754427195 | 1.47326E−02 | 0.745120887406124 | 0.000431187 | 0.7446896730961782 | 2.73825E−08 | 0.744689700478633 |
| 0.6 | 0.7153744093633583 | 1.55140E−02 | 0.731317887808574 | 0.000429485 | 0.7308883728944706 | 2.98840E−08 | 0.730888402778508 |
| 0.7 | 0.7090695033855021 | 1.61818E−02 | 0.725674565817607 | 0.000423267 | 0.7252512673367656 | 3.19353E−08 | 0.725251299272098 |
| 0.8 | 0.7102098229998883 | 1.68173E−02 | 0.727442024142068 | 0.000414938 | 0.7270270524626191 | 3.37550E−08 | 0.727027086217657 |
| 0.9 | 0.7182708868437886 | 1.74727E−02 | 0.736149394556421 | 0.000405806 | 0.735743553051839 | 3.54931E−08 | 0.735743588544958 |
| 1.0 | 0.7329587357703423 | 1.81816E−02 | 0.751536906604233 | 0.000396555 | 0.7511403147092988 | 3.72487E−08 | 0.7511403519579868 |

***2(b)***

***2(c)***

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Euler’s Method |  | Euler’s Modified Method |  | Runge-Kutta Method |  | Exact Solution |
| *xn* |  | ER |  | ER | ER |  |  |
| 0.1 | 0.9104875290532907 | 3.02160E−03 | 0.9137386 3169801 | 0.000229504 | 0.9135091276397532 | 2.59029E−10 | 0.913509127898782 |
| 0.2 | 0.8443847744779372 | 4.83374E−03 | 0.849562976531398 | 0.000344458 | 0.8492185180509749 | 6.51469E−10 | 0.849218518702443 |
| 0.3 | 0.7958587285640485 | 5.96467E−03 | 0.802223388565394 | 0.000399991 | 0.8018233969598182 | 9.97784E−10 | 0.8018233979576023 |
| 0.4 | 0.7610777171130718 | 6.70587E−03 | 0.768207590571715 | 0.000424004 | 0.7677835848899629 | 1.26955E−09 | 0.7677835861595071 |
| 0.5 | 0.7374639944962212 | 7.22571E−03 | 0.745120887406124 | 0.000431187 | 0.7446896989999765 | 1.47866E−09 | 0.7446897004786337 |
| 0.6 | 0.7232631491061028 | 7.62525E−03 | 0.731317887808574 | 0.000429485 | 0.7308884011344937 | 1.64401E−09 | 0.7308884027785085 |
| 0.7 | 0.7172840690076906 | 7.96723E−03 | 0.725674565817607 | 0.000423267 | 0.7252512974898684 | 1.78223E−09 | 0.7252512992720983 |
| 0.8 | 0.7187354764318908 | 8.29161E−03 | 0.727442024142068 | 0.000414938 | 0.7270270843117828 | 1.90588E−09 | 0.7270270862176577 |
| 0.9 | 0.7271193149751998 | 8.62427E−03 | 0.736149394556421 | 0.000405806 | 0.7357435865210894 | 2.02387E−09 | 0.7357435885449581 |
| 1.0 | 0.7421585135282368 | 8.98184E−03 | 0.751536906604233 | 0.000396555 | 0.7511403498157103 | 2.14228E−09 | 0.7511403519579868 |

***2(d)***

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Euler’s Method |  | Euler’s Modified Method |  | Runge-Kutta Method |  | Exact Solution |
| *xn* |  | ER |  | ER |  | ER |  |
| 0.1 | 0.9120236443372686 | 1.48548E−03 | 0.9135226 4673271 | 1.35188E-05 | 0.9135091278869912 | 1.17910E−11 | 0.913509127898782 |
| 0.2 | 0.846835976864134 | 2.38254E−03 | 0.849238872455001 | 2.03538E-05 | 0.8492185186679586 | 3.44851E−11 | 0.849218518702443 |
| 0.3 | 0.7988774812685904 | 2.94592E−03 | 0.801847096047129 | 2.36981E-05 | 0.8018233979021255 | 5.54771E−11 | 0.8018233979576023 |
| 0.4 | 0.7644662941812183 | 3.31729E−03 | 0.767808764293526 | 2.51781E-05 | 0.7677835860871474 | 7.23600E−11 | 0.7677835861595071 |
| 0.5 | 0.7411106710960523 | 3.57903E−03 | 0.744715355453644 | 2.5655E-05 | 0.7446897003930565 | 8.55770E−11 | 0.7446897004786337 |
| 0.6 | 0.7271075635837666 | 3.78084E−03 | 0.730913999935655 | 2.55972E-05 | 0.7308884026823441 | 9.61640E−11 | 0.7308884027785085 |
| 0.7 | 0.721297592738653 | 3.95371E−03 | 0.725276562956256 | 2.52637E-05 | 0.7252512991670091 | 1.05089E−10 | 0.7252512992720983 |
| 0.8 | 0.722909634547887 | 4.11745E−03 | 0.727051884359835 | 2.47981E-05 | 0.7270270861045542 | 1.13103E−10 | 0.7270270862176577 |
| 0.9 | 0.7314586485260627 | 4.28494E−03 | 0.735767867647858 | 2.42791E-05 | 0.7357435884242071 | 1.20751E−10 | 0.7357435885449581 |
| 1.0 | 0.7466759122017882 | 4.46444E−03 | 0.751164100109758 | 2.37482E-05 | 0.751140351829581 | 1.28405E−10 | 0.7511403519579868 |

***SEGMENT:5 DISCUSSION OF RESULTS***

The results which are obtained shown in Tables 1(a)-(d) and Tables 2(a)-(d). The numerical approximate solution is calculated with step sizes 0.1, 0.05, 0.025 and 0.0125 and utmost errors also are evaluated at particular step size. From the tables for each technique we state that a numerical solution converges to the accurate solution if the step size leads to decreased errors such that in the limit when the step size to zero the errors also tends to zero. We saw that the Euler method, Euler modified method iterations using the step size 0.1 and 0.05 didn’t converge to exact solution but for step size 0.025 and 0.0125 converge gradually to exact solution. Also we observed that the Runge-Kutta approximations for same step size converge firstly to exact solution. This shows that the small step size gives the improved estimation. The Runge-Kutta method of order four requires four evaluations per step, so it should give more accurate results in compare with Euler method, Euler modified method with one-fourth the step size if it is to be superior. Lastly we observed that the fourth order Runge-Kutta method is converging quicker than the Euler method, Euler modified method and it is the best efficient method for solving initial value problems for ordinary differential equations.

***SEGMENT: 6 CONCLUSION***

In this paper, Euler improved, Euler modified method and Runge-Kutta method are used for solving ordinary differential equation (ODE) in initial value problems (IVP). Finding more accurate results needs the step size smaller for all methods. From the figures we can see the accuracy of the methods for decreasing the step size h. The numerical solutions obtained by the three proposed methods are in good agreement with exact solutions. Comparing the results of the three methods under investigation, we observed that the rate of convergence of Euler improved, Euler modified method is O(h) and the rate of convergence of fourth-order Runge Kutta method is O . The Euler improved, Euler modified method was found to be less accurate due to the inaccurate numerical results that were obtained from the approximate solution in comparison to the exact solution. From the study the Runge-Kutta method was found to be generally more accurate and also the approximate solution converged faster to the exact solution when compared to the Euler improved, Euler modified method. It may be concluded that the Runge Kutta method is powerful and more efficient in finding numerical solutions of initial value problems (IVP).

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